

# OPERATOR ALGEBRAS WITH CONTRACTIVE APPROXIMATE IDENTITIES: WEAK COMPACTNESS AND THE SPECTRUM

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**ABSTRACT.** We continue our study of operator algebras with contractive approximate identities (cais) by presenting a couple of interesting examples of operator algebras with cais, which in particular answer questions raised in previous papers in this series, for example about whether, roughly speaking, ‘weak compactness’ of an operator algebra, or the lack of it, can be seen in the spectra of its elements.

## 1. INTRODUCTION

An *operator algebra* is a closed subalgebra of  $B(H)$ , for a Hilbert space  $H$ . An operator algebra with a contractive approximate identity (cai) is called *approximately unital*. Here we construct an interesting new approximately unital operator algebra, and use it to solve questions arising in our earlier work, for example about whether, roughly speaking, ‘weak compactness’ of an operator algebra, or the lack of it, can be seen in the spectra of its elements. We now describe some background for this. We recall that a semisimple Banach algebra  $A$  is a modular annihilator algebra iff no element of  $A$  has a nonzero limit point in its spectrum [12, Theorem 8.6.4]. If  $A$  is also commutative then this is equivalent to the Gelfand spectrum of  $A$  being discrete [11, p. 400]. We write  $M_{a,b} : A \rightarrow A : x \mapsto axb$ , where  $a, b \in A$ . Recall that a Banach algebra is *compact* if the map  $M_{a,a}$  is compact for all  $a \in A$ . We say that  $A$  is *weakly compact* if  $M_{a,a}$  is weakly compact for all  $a \in A$ . If  $A$  is approximately unital and commutative then  $A$  is weakly compact iff  $A$  is an ideal in its bidual  $A^{**}$  (see e.g. [10, 1.4.13]). In the noncommutative case  $A$  is weakly compact iff  $A$  is a *hereditary subalgebra* (or *HSA*) in its bidual (see [2, Lemma 5.1]). It is known [12] that every compact semisimple Banach algebra is a modular annihilator algebra (and conversely every semisimple ‘annihilator algebra’, or more generally any Banach algebra with dense socle, is compact). Thus it is of interest to know if there are any connections for operator algebras between being a semisimple modular annihilator algebra, and being weakly compact. See the discussion after Proposition 5.6 in [2], where some specific questions along these lines are raised. We have solved these here; indeed we have by now solved essentially all open questions posed in our previous papers [7, 8, 2]. In particular we show here, first, that a semisimple approximately unital operator algebra which is a modular annihilator algebra need not be weakly compact, nor need it be nc-discrete. (The latter term will be defined before Corollary 2.13, when it is needed.) Second, an approximately unital commutative weakly compact semisimple operator algebra  $A$  need not have

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countable or scattered spectrum (in fact the spectrum of some of its elements can have nonempty interior).

2. A SEMISIMPLE OPERATOR ALGEBRA WHICH IS A MODULAR ANNIHILATOR ALGEBRA BUT IS NOT WEAKLY COMPACT

Let  $(c_n)$  be an unbounded increasing sequence in  $(0, \infty)$ . For each  $n \in \mathbb{N}$  let  $d_n$  be the diagonal matrix in  $M_n$  with  $c_n^k$  as the  $k$ th diagonal entry. If  $M$  is the von Neumann algebra  $\oplus_n^\infty (M_n \oplus M_n)$ , we let  $N$  be its weak\*-closed unital subalgebra consisting of tuples  $((x_n, d_n x_n d_n^{-1}))$ , for all  $(x_n) \in \oplus_n^\infty M_n$ . We define  $A_{00}$  to be the finitely supported tuples in  $N$ , and  $A_0$  to be the closure of  $A_{00}$ . That is,  $A_0$  is the intersection of the  $c_0$ -sum  $C^*$ -algebra  $\oplus_n^\infty (M_n \oplus M_n)$  with  $N$ . We sometimes simply write  $(x_n)$  for the associated tuple in  $N$ .

**Lemma 2.1.** *Let  $A$  be any closed subalgebra of  $N$  containing  $A_0$ . Then  $A$  is semisimple.*

*Proof.* For any nonzero  $x = (x_n) \in A$ , choose  $m$  and  $i$  with  $z = x_m e_i \neq 0$ , where  $(e_i)$  is the usual basis of  $\mathbb{C}^m$ . Choose  $y_m \in M_m$  with  $y_m z = e_i$ , and otherwise set  $y_n = 0$ . Then  $y = (y_n) \in A_0$ , and the copy of  $e_i$  is in the kernel of  $I - yx$ . Hence  $I - yx$  is not invertible in  $A^1$ , and so  $x$  is not in the Jacobson radical by a well known characterization of that radical. Thus  $A$  is semisimple.  $\square$

Endow  $M_n$  with a norm  $p_n(x) = \max\{\|x\|, \|d_n x d_n^{-1}\|\}$ . Then  $N \cong \oplus_n^\infty (M_n, p_n(\cdot))$  isometrically, and we write  $p(\cdot)$  for the norm on the latter space, so  $p((x_n)) = \sup_n p_n(x_n)$ . We sometimes view  $p$  as the norm on  $N$  via the above identification. Let  $L_n$  be the left shift on  $\mathbb{C}^n$ , so that in particular  $L_n e_1 = 0$ . Note that  $d_n L_n d_n^{-1} = \frac{1}{c_n} L_n$ , and that  $p_n(L_n) = 1$  if  $n \geq 2$ . For  $n, k \in \mathbb{N}$  with  $n \geq k$  define an ‘integer interval’  $E_{n,k} = \mathbb{N}_0 \cap [\frac{n}{k}, \frac{2n}{k}]$ . Set  $\mu_{n,k} = |E_{n,k}|$  if  $n \geq k$ , with  $\mu_{n,k} = 1$  if  $n < k$ . Then  $\mu_{n,k}$  is strictly positive for all  $n, k$ . For  $n \geq k$  define  $u_{n,k} = \frac{1}{\mu_{n,k}} \sum_{i \in E_{n,k}} (L_n)^i \in M_n$ . If  $n < k$  set  $u_{n,k} = I_n$ . Define  $u_k = (u_{n,k})_{n \in \mathbb{N}}$ . We have

$$p_n(u_{n,k}) \leq \max_{i \in E_{n,k}} p((L_n)^i) \leq 1, \quad n \geq k,$$

and so

$$p(u_k) \leq 1, \quad k \in \mathbb{N}.$$

The operator algebra we are interested in is

$$A = \{a \in N : p(au_k - a) + p(u_k a - a) \rightarrow 0\}.$$

This will turn out to be the largest subalgebra of  $N$  having  $(u_k)$  as a cai. First, a preliminary estimate:

**Lemma 2.2.** *Let  $L \in \text{Ball}(B)$  for a Banach algebra  $B$ . Suppose that  $E_1$  is a set of  $\mu_1$  integers from  $[0, n]$ , and  $E_2$  is a set of  $\mu_2$  consecutive nonnegative integers. If  $u_i = \frac{1}{\mu_i} \sum_{i \in E_i} L^i$  then*

$$\|u_1 u_2 - u_2\| \leq \frac{2n}{\mu_2}.$$

*Proof.* If  $n \geq \mu_2$  then

$$\|u_1 u_2 - u_2\| \leq \|u_1\| \|u_2\| + \|u_2\| \leq 2 \leq \frac{2n}{\mu_2}.$$

So we may assume that  $n < \mu_2$ . Let  $m_0 = \min E_2$ . Then

$$u_1 u_2 = \frac{1}{\mu_1 \mu_2} \sum_{j \in E_1, k \in E_2} L^{j+k} = \sum_{m_0 \leq m < m_0 + n + \mu_2} \lambda_m L^m,$$

where  $\lambda_m$  is  $\frac{1}{\mu_1 \mu_2}$  times the number of pairs in  $E_1 \times E_2$  which sum to  $m$ . Since

$$\mu_1 \leq n + 1 \leq \mu_2,$$

and since the number of such pairs cannot exceed  $\mu_1 = |E_1|$ , we have

$$0 \leq \lambda_m \leq \frac{1}{\mu_2}.$$

If  $m \in [m_0 + n, m_0 + \mu_2)$  then  $m - k \in E_2$  for any integer  $k$  in  $[0, n]$ , and so  $m - E_1 \subset E_2$ . We deduce that

$$\lambda_m = \frac{1}{\mu_2}, \quad m \in [m_0 + n, m_0 + \mu_2).$$

Since  $u_2 = \frac{1}{\mu_2} \sum_{m_0 \leq m < m_0 + \mu_2} L^m$  we have

$$u_1 u_2 - u_2 = \sum_{m_0 \leq m < m_0 + n} \left( \lambda_m - \frac{1}{\mu_2} \right) L^m + \sum_{m_0 + \mu_2 \leq m < m_0 + n + \mu_2} \lambda_m L^m.$$

No coefficient in the last sum has modulus greater than  $\frac{1}{\mu_2}$ , and there are  $2n$  nonzero coefficients, so

$$\|u_1 u_2 - u_2\| \leq \frac{2n}{\mu_2} \max_m \|L^m\| = \frac{2n}{\mu_2}$$

as desired.  $\square$

**Corollary 2.3.** *Let  $A = \{a \in N : p(au_k - a) + p(u_k a - a) \rightarrow 0\}$ . Then  $A$  is a semisimple operator algebra with cai  $(u_k)$ , and  $A_0$  is an ideal in  $A$ .*

*Proof.* We first show  $u_r \in A$  for all  $r \in \mathbb{N}$ . Let  $k \geq r$ . If  $n \geq k$  then  $E_{n,k}$  is a subset of  $[0, \frac{2n}{k}]$ , and  $\mu_{n,k}$  is either  $\lfloor \frac{n}{k} \rfloor$  or  $\lfloor \frac{n}{k} \rfloor + 1$ . By Lemma 2.2, we have

$$p_n(u_{n,k} u_{n,r} - u_{n,r}) \leq \frac{2 \lfloor \frac{2n}{k} \rfloor}{\lfloor \frac{n}{r} \rfloor}, \quad r \geq n \geq k.$$

If  $n < k$  then  $p_n(u_{n,k} u_{n,r} - u_{n,r}) = 0$ . If  $k \geq 2tr$  for an integer  $t > 1$  then

$$\frac{2 \lfloor \frac{2n}{k} \rfloor}{\lfloor \frac{n}{r} \rfloor} \leq \frac{\lfloor \frac{n}{tr} \rfloor}{\lfloor \frac{n}{r} \rfloor} \leq \frac{1}{t}.$$

Thus  $p_n(u_{n,k} u_{n,r} - u_{n,r}) \leq \frac{2}{t}$  for  $k \geq 2tr$ , so

$$p(u_k u_r - u_r) = \sup_n p_n(u_{n,k} u_{n,r} - u_{n,r}) \leq \frac{2}{t}, \quad k \geq 2tr.$$

So  $u_k u_r \rightarrow u_r$  with  $k$ , and so  $u_r \in A$  for all  $r \in \mathbb{N}$ .

It is now obvious that  $A$ , being a subalgebra of the operator algebra  $N$ , is an operator algebra with cai  $(u_k)$ . It is elementary that for any matrix  $x$  in the copy  $M'_n$  of  $M_n$  in  $A_0$  we have  $x u_k \rightarrow x$  and  $u_k x \rightarrow x$ , since for example  $u_k x = x$  for  $k > n$ . Hence  $A_0 \subset A$ , so that  $A$  is semisimple by Lemma 2.1. Since  $M'_n$  is an ideal in  $N$ , so is  $A_0$ , giving the last statement.  $\square$

In the following result, and elsewhere,  $\|\cdot\|$  denotes the usual norm on  $M_n$  or on  $\oplus_n^\infty M_n$ .

**Lemma 2.4.** *For each  $n \in \mathbb{N}$  and  $k \leq n$ , we have  $\|u_{n,k}\| \geq 1 - \frac{2}{k}$  and  $\|u_{n,k}^3\| \geq 1 - \frac{6}{k}$ .*

*Proof.* If  $\eta$  is the unit vector  $(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$  in  $\mathbb{C}^n$ , then it is easy to see that

$$\langle (L_n)^k \eta, \eta \rangle = 1 - \frac{k}{n}, \quad 0 \leq k \leq n.$$

Since  $u_{n,k}$  is an average of powers  $(L_n)^j$  with  $0 \leq j \leq \frac{2n}{k}$ , we have

$$\langle u_{n,k} \eta, \eta \rangle \geq 1 - \frac{\frac{2n}{k}}{n} = 1 - \frac{2}{k}.$$

Similarly,  $u_{n,k}^3$  is a weighted average of powers  $(L_n)^j$  with  $0 \leq j \leq \frac{6n}{k}$ .  $\square$

We note that the diagonal matrix units  $e_{i,i}^n$  are orthogonal projections, and are also minimal idempotents in  $A$  (that is, have the property that  $eAe = \mathbb{C}e$ ).

**Theorem 2.5.**  *$A$  is not weakly compact, and is not separable.*

*Proof.* Note that  $A$  is an  $\ell^\infty$ -bimodule via the action

$$(\alpha_n) \cdot (T_n) = (T_n) \cdot (\alpha_n) = (\alpha_n T_n), \quad (\alpha_n) \in \ell^\infty, (T_n) \in A.$$

We will use this to embed  $\ell^\infty$  isomorphically in  $xAx$ , where  $x = u_r$  for large enough  $r$ . Note that

$$\ell^\infty \cdot x^3 = x(\ell^\infty \cdot x)x \subset xAx.$$

Choosing  $r$  with  $1 - \frac{6n}{r} \geq \frac{1}{2}$ , we have that  $\|u_{n,r}^3\| \geq \frac{1}{2}$  for all  $n \in \mathbb{N}$  (recall  $u_{n,r} = I$  if  $n < r$ ). Thus for  $\vec{\alpha} = (\alpha_n) \in \ell^\infty$  we have

$$p(\vec{\alpha} \cdot x^3) \geq \|\vec{\alpha} \cdot x^3\| = \|\vec{\alpha} \cdot u_r^3\| = \sup_n |\alpha_n| \|u_{n,r}^3\| \geq \frac{1}{2} \sup_n |\alpha_n|,$$

and so the map  $\vec{\alpha} \mapsto \vec{\alpha} \cdot x^3$  is a bicontinuous injection of  $\ell^\infty$  into  $xAx$ . Thus  $A$  is not weakly compact, nor separable.  $\square$

**Lemma 2.6.** *If  $T = (T_n) \in A$ , then  $\|d_n T_n d_n^{-1}\| \rightarrow 0$  as  $n \rightarrow \infty$ . Thus the spectral radius  $r(T_n) \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof.* Given  $\epsilon > 0$  there exists an  $m \in \mathbb{N}$  such that

$$p_n(u_{n,m} T_n - T_n) + p_n(T_n u_{n,m} - T_n) < \frac{\epsilon}{2} p(u_m T - T) + p(T u_m - T) < \frac{\epsilon}{2}, n \in \mathbb{N}.$$

We have noted that  $d_n L_n d_n^{-1} = \frac{1}{c_n} L_n$ , and for  $n \geq m$  the operator  $u_{n,m}$  is an average of powers  $L_n^j$ , so for  $n \geq m$  we have

$$\|d_n u_{n,m} d_n^{-1}\| \leq \max_{j \in \mathbb{N}} \|d_n L_n^j d_n^{-1}\| \leq \frac{1}{c_n}.$$

Thus

$$\|d_n T_n u_{n,m} d_n^{-1}\| \leq \frac{1}{c_n} \|d_n T_n d_n^{-1}\| \leq \frac{1}{c_n} p(T).$$

Consequently, for  $n \geq m$  the quantity  $\|d_n T_n d_n^{-1}\|$  is dominated by

$$\|d_n (T_n u_{n,m} - T_n) d_n^{-1}\| + \|d_n T_n u_{n,m} d_n^{-1}\| \leq p_n(T_n u_{n,m} - T_n) + \frac{1}{c_n} p(T) \leq \frac{\epsilon}{2} + \frac{1}{c_n} p(T).$$

The result is clear from this.  $\square$

For a matrix  $B$  write  $\overline{\Delta}_U B$  for the upper triangular projection of  $B$  (that is, we change  $b_{ij}$  to 0 if  $i > j$ ). Similarly, write  $\Delta_L B$  for the strictly lower triangular part of  $B$ . In the next results, as usual  $\binom{r}{s} = 0$  if  $0 \leq r < s$  are integers.

**Lemma 2.7.** *If  $0 \neq T = (T_n) \in A$ , and  $\epsilon > 0$  is given, there exist  $k, m \in \mathbb{N}$  such that for all  $r \in \mathbb{N}_0$  and  $n \geq \max\{k, m\}$ , we have*

$$\|(\overline{\Delta}_U T_n)^r\| \leq \sum_{s=0}^{k-1} \binom{r}{s} (2p(T))^r \epsilon^{r-s}.$$

*Proof.* The  $i$ - $j$  entry  $T_{n,i,j}$  of  $T_n$  equals  $\langle T_n e_j, e_i \rangle = c_n^{j-i} \langle d_n T_n d_n^{-1} e_j, e_i \rangle$ , and so

$$|T_{n,i,j}| = c_n^{j-i} |\langle d_n T_n d_n^{-1} e_j, e_i \rangle| \leq c_n^{j-i} p_n(T_n), \quad T = (T_n) \in A.$$

It follows from this that

$$\left\| \sum_{j=1}^{n-r} T_{n,j+r,j} E_{j+r,j} \right\| = \max_{j \leq n-r} |T_{n,j+r,j}| \leq c_n^{-r} p_n(T_n),$$

if  $r < n$ . Since  $\sum_{r=1}^{n-1} (\sum_{j=1}^{n-r} T_{n,j+r,j} E_{j+r,j}) = \Delta_L T_n$ , we deduce that

$$(2.1) \quad \|\Delta_L T_n\| = \|T_n - \overline{\Delta}_U T_n\| \leq \sum_{r=1}^{n-1} c_n^{-r} p_n(T_n) \leq \frac{p_n(T_n)}{c_n - 1} \leq \frac{p(T)}{c_n - 1}.$$

Given  $\epsilon > 0$  choose  $k$  with  $p(u_k T - T) < \epsilon p(T)$ , and let  $n \geq k$ . Then

$$\|u_{n,k} T_n - T_n\| \leq p_n(u_{n,k} T_n - T_n) < \epsilon p(T),$$

and so

$$\|u_{n,k} \overline{\Delta}_U T_n - \overline{\Delta}_U T_n\| \leq \epsilon p(T) + \|(u_{n,k} - I)(T_n - \overline{\Delta}_U T_n)\| \leq p(T) \left( \epsilon + \frac{2}{c_n - 1} \right),$$

since

$$u_{n,k} \overline{\Delta}_U T_n - \overline{\Delta}_U T_n = (I - u_{n,k})(T_n - \overline{\Delta}_U T_n) + (u_{n,k} T_n - T_n).$$

Let  $S_1 = u_{n,k} \overline{\Delta}_U T_n$  and  $S_2 = \overline{\Delta}_U T_n - S_1$ , then  $\|S_2\| \leq p(T) \left( \epsilon + \frac{2}{c_n - 1} \right)$ , by the last displayed equation. Also,

$$\|S_1\| \leq \|\overline{\Delta}_U T_n\| \leq p(T) + \|(I - \overline{\Delta}_U) T_n\| \leq p(T) + \frac{p(T)}{c_n - 1} = p(T) \frac{c_n}{c_n - 1}$$

by (2.1).

Now  $\overline{\Delta}_U T_n = S_1 + S_2$ , so  $(\overline{\Delta}_U T_n)^r$  is a sum from  $s = 0$  to  $r$ , of  $\binom{r}{s}$  times terms which are a product of  $r$  factors,  $s$  of which are  $S_1$  and  $r - s$  of which are  $S_2$ . Note that any product of upper triangular  $n \times n$  matrices that has  $k$  or more factors which equal  $S_1$ , is zero. This is because multiplication of an upper triangular matrix  $U$  by  $u_{n,k}$  (and hence by  $S_1$ ) decreases the number of nonzero ‘superdiagonals’ of  $B$  by a number  $\geq \frac{n}{k}$ , so after  $k$  such multiplications we are left with the zero matrix. Thus we can assume that  $s < k$  above. Using the estimates at the end of the last paragraph, we deduce that

$$\|(S_1 + S_2)^r\| \leq \sum_{s=0}^{k-1} \binom{r}{s} \|S_1\|^s \|S_2\|^{r-s} \leq \sum_{s=0}^{k-1} \binom{r}{s} \left( p(T) \frac{c_n}{c_n - 1} \right)^s \left( p(T) \left( \epsilon + \frac{2}{c_n - 1} \right) \right)^{r-s}.$$

Since  $c_n \rightarrow \infty$  we may choose  $m$  such that  $\frac{c_n}{c_n - 1} < 2$  and  $\epsilon + \frac{2}{c_n - 1} < 2\epsilon$  for all  $n \geq m$ . Thus for  $n \geq \max\{k, m\}$ , we have

$$\|(\overline{\Delta}_U T_n)^r\| = \|(S_1 + S_2)^r\| \leq \sum_{s=0}^{k-1} \binom{r}{s} (2p(T))^r \epsilon^{r-s}$$

as desired.  $\square$

For  $k \in \mathbb{N}$  and positive numbers  $b, \epsilon$ , define a quantity  $K(k, b, \epsilon) = \frac{1}{2b(1-\epsilon)\epsilon^k}$ .

**Lemma 2.8.** *If  $0 \neq T = (T_n) \in A$ , and  $\epsilon > 0$  is given, there exist  $k, m \in \mathbb{N}$  such that for all  $\lambda \in \mathbb{C}$  with  $|\lambda| > 4p(T)\epsilon$ , and  $n \geq \max\{k, m\}$ , we have  $\lambda I - \overline{\Delta}_U T_n$  and  $\lambda I - T_n$  invertible in  $M_n$ , and both*

$$\|(\lambda I - \overline{\Delta}_U T_n)^{-1}\| \leq K(k, p(T), \epsilon)$$

and

$$\|(\lambda I - T_n)^{-1}\| \leq 2K(k, p(T), \epsilon).$$

*Proof.* If  $|\lambda| > 2p(T)\epsilon$  then

$$\sum_{r=0}^{\infty} \|\lambda^{-r-1} (\overline{\Delta}_U T_n)^r\| \leq |\lambda|^{-1} \sum_{r=0}^{\infty} \sum_{s=0}^{k-1} \binom{r}{s} \left(\frac{2p(T)}{|\lambda|}\right)^r \epsilon^{r-s},$$

by Lemma 2.7, for  $n \geq \max\{k, m\}$ , where  $k, m$  are as in that lemma. However the latter quantity equals

$$|\lambda|^{-1} \sum_{s=0}^{k-1} \sum_{r=0}^{\infty} \binom{r}{s} \left(\frac{2p(T)\epsilon}{|\lambda|}\right)^{r-s} \left(\frac{2p(T)}{|\lambda|}\right)^s = |\lambda|^{-1} \sum_{s=0}^{k-1} \left(\frac{2p(T)}{|\lambda|}\right)^s \left(1 - \frac{2p(T)\epsilon}{|\lambda|}\right)^{-s-1}$$

using the binomial formula. This is finite, so  $\sum_{r=0}^{\infty} \lambda^{-r-1} (\overline{\Delta}_U T_n)^r$  converges, and this is clearly an inverse for  $\lambda I - \overline{\Delta}_U T_n$ . If  $|\lambda| > 4p(T)\epsilon$ , then the sum in the last displayed equation is dominated by

$$\frac{1}{4p(T)\epsilon} \sum_{s=0}^{k-1} \left(\frac{1}{2\epsilon}\right)^s 2^{s+1} = \frac{1}{2p(T)(1-\epsilon)} \frac{1-\epsilon^k}{\epsilon^k} \leq K(k, p(T), \epsilon).$$

We also obtain

$$(2.2) \quad \|(\lambda I - \overline{\Delta}_U T_n)^{-1}\| \leq K(k, p(T), \epsilon).$$

By increasing  $m$  if necessary, we can assume that  $c_n - 1 > 2p(T)K(k, p(T), \epsilon)$ . Then by (2.1) we have

$$\|T_n - \overline{\Delta}_U T_n\| \leq \frac{p(T)}{c_n - 1} < \frac{1}{2K(k, p(T), \epsilon)}.$$

A simple consequence of the Neumann lemma is that if  $R$  is invertible and  $\|H\| < \frac{1}{2\|R^{-1}\|}$ , then  $R + H$  is invertible and  $\|(R + H)^{-1}\| \leq 2\|R^{-1}\|$ . Setting  $R = \lambda I - \overline{\Delta}_U T_n$  and  $H = \overline{\Delta}_U T_n - T_n$ , we have

$$\|H\| < \frac{1}{2K(k, p(T), \epsilon)} < \frac{1}{2\|R^{-1}\|}$$

by (2.2). Hence  $R + H = \lambda I - T_n$  is invertible, and by (2.2) again the norm of its inverse is dominated by  $2\|R^{-1}\| \leq 2K(k, p(T), \epsilon)$ .  $\square$

The quantity  $K(k, p(T), \epsilon)$  above is independent of  $n$ , which gives:

**Corollary 2.9.** *The spectrum of every element of  $A$  is finite or a null sequence and zero. Hence  $A$  is a modular annihilator algebra.*

*Proof.* Let  $0 \neq T = (T_n) \in A$ . We will show that the spectrum of  $T$  is finite or a null sequence and zero. It is sufficient to show that if  $\epsilon > 0$  is given, there exists  $m_0 \in \mathbb{N}$  such that if  $|\lambda| > 4p(T)\epsilon$ , and if  $\lambda$  is not in the spectrum of  $T_1, \dots, T_{m_0}$ , then  $\lambda \notin \text{Sp}_A(T)$ . So assume these conditions, and let  $m_0 = \max\{k, m\}$  as in Lemma 2.8. For  $n \geq m_0$  we have by Lemma 2.8 that  $\lambda I - T_n$  is invertible, and the usual matrix norm of its inverse is bounded independently of  $n$ . By assumption this is also true for  $n < m_0$ . By Lemma 2.6 there is a  $q$  such that  $\|d_n T_n d_n^{-1}\| < \epsilon$  for  $n \geq q$ . If  $|\lambda| > \epsilon$  then  $(\lambda I - T_n)^{-1} = \sum_{r=0}^{\infty} \lambda^{-r-1} T_n^r$  and

$$\|d_n(\lambda I - T_n)^{-1} d_n^{-1}\| = \left\| \sum_{r=0}^{\infty} \lambda^{-r-1} d_n T_n^r d_n^{-1} \right\| \leq \sum_{r=0}^{\infty} |\lambda|^{-r-1} \epsilon^r = |\lambda|^{-1} \left(1 - \frac{\epsilon}{|\lambda|}\right)^{-1}.$$

Thus  $(p_n((\lambda I - T_n)^{-1}))$  is bounded independently of  $n$ . Hence  $((\lambda I - T_n)^{-1}) \in N$ , and this is an inverse in  $N$  for  $\lambda I - T$ . Thus the spectrum of  $T$  in  $N$  is finite or a null sequence and zero. The spectrum in  $A$  might be bigger, but since the boundary of its spectrum cannot increase,  $\text{Sp}_A(T)$  is also finite or a null sequence and zero.

The last statements follow from [12, Chapter 8].  $\square$

We point out some more features of our example  $A$ , in hope that these may further its future use as a counterexample in the subject.

We recall that the multiplier algebra  $M(A)$  of  $A$  is identified with the idealizer of  $A$  in its bidual  $A^{**}$  (that is, the set of elements  $\alpha \in A^{**}$  such that  $\alpha A \subset A$  and  $A\alpha \subset A$ ). It can also be viewed as the idealizer of  $A$  in  $B(H)$ , if  $A$  is represented nondegenerately and completely isometrically on a Hilbert space  $H$ . See [4, Section 2.6] for this.

**Proposition 2.10.** *The multiplier algebra of  $A$  may be taken to be  $\{x \in N : xA + Ax \subset A\}$ . This is also valid with  $N$  replaced by  $M$ .*

*Proof.* Viewing  $M = \oplus_n^\infty (M_n \oplus M_n)$  as represented on  $H = \oplus_n^2 (\mathbb{C}^n \oplus \mathbb{C}^n)$ , it is clear that  $D_0$ , and hence also  $A$ , acts nondegenerately on  $H$ . So the multiplier algebra  $M(A)$  may be viewed as a subalgebra of  $B(H)$ . We also see that the weak\* continuous extension  $\tilde{\pi} : A^{**} \rightarrow N$  of the ‘identity map’ on  $A$ , is a completely isometric homomorphism from the copy of  $M(A)$  in  $A^{**}$  onto the copy of  $M(A)$  in  $B(H)$ , and in particular, the latter is contained in  $N$ . So the latter is  $M(A) = \{x \in N : xA + Ax \subset A\}$ . A similar argument works with  $M$  replaced by  $N$ .  $\square$

We note that if  $D_n$  is the commutative diagonal  $C^*$ -algebra in  $M_n$ , then there is a natural isometric copy  $D$  of  $\oplus_n^\infty D_n$  inside  $N$ , namely the tuples  $((x_n, x_n))$  for a bounded sequence  $x_n \in D_n$ .

We assume henceforth that  $c_n > 1$  for all  $n$ .

In the next results  $\Delta(A)$  denotes the ‘diagonal’  $A \cap A^*$  of  $A$  (here  $A^*$  is the set of ‘adjoint operators’ (or ‘involutions’) of elements in  $A$ ). See 2.1.2 in [4].

**Proposition 2.11.** *The diagonal  $\Delta(A)$  equals the natural copy  $D_0$  of the  $c_0$ -sum  $C^*$ -algebra  $\oplus_n^\infty D_n$  inside  $A$ .*

*Proof.* If  $((x_n, d_n x_n d_n^{-1}))$  is selfadjoint, then  $x_n$  is selfadjoint, and  $d_n x_n d_n^{-1}$  is selfadjoint, which forces  $d_n^2$  to commute with  $x_n$ . However this implies that  $x_n$  is diagonal. Since  $\Delta(N) = N \cap N^*$  is spanned by its selfadjoint elements it follows that  $\Delta(N) = D$ . Therefore  $\Delta(A) = D \cap A$ , and this contains  $D_0$  since  $D_0 \subset A_0 \subset A$  by Corollary 2.3. The reverse containment follows easily from Lemma 2.6, but we

give a shorter proof. Let  $(a_n) \in D \cap A$ , with  $a_n \in D_n$  for each  $n$ . If  $\epsilon > 0$  is given, choose  $k$  such that  $p(u_k(a_n) - (a_n)) < \epsilon$ . Choose  $m$  with  $u_{n,k}$  strictly upper triangular for all  $n \geq m$ . Then for  $n \geq m$  we have  $|a_n(i)|$ , which is the modulus of the  $i$ - $i$  entry of  $(u_k(a_n) - (a_n))$ , is dominated by

$$\|u_{n,k} a_n - a_n\| \leq p(u_k(a_n) - (a_n)) < \epsilon.$$

Thus  $\|a_n\| < \epsilon$  for  $n \geq m$ , so that  $(a_n) \in D_0$ .  $\square$

We recall some notation from e.g. [4, Chapter 2] and [5]. By a *projection* we mean an orthogonal projection. The second dual  $A^{**}$  is also an operator algebra with its (unique) Arens product, this is also the product inherited from the von Neumann algebra  $B^{**}$  if  $A$  is a subalgebra of a  $C^*$ -algebra  $B$ . Note that  $A^{**}$  has an identity  $1_{A^{**}}$  of norm 1 since  $A$  has a cai. We say that a projection  $p \in A^{**}$  is an *open projection* if there is a net  $x_t \in A$  with  $x_t = px_t \rightarrow p$  weak\*, or equivalently with  $x_t = px_t p \rightarrow p$  weak\* (see [5, Theorem 2.4]). These are also the open projections  $p$  in the sense of Akemann [1] in  $B^{**}$ , where  $B$  is a  $C^*$ -algebra containing  $A$ , such that  $p \in A^{\perp\perp}$ . The complement  $p^\perp = 1_{A^{**}} - p$  of an open projection for  $A$  is called a *closed projection* for  $A$ .

**Corollary 2.12.** *Projections in  $A^{**}$  which are both open and closed, or equivalently (by [5, Example 2.1] and the first lines of the proof of [3, Proposition 2.12]) which are in the multiplier algebra  $M(A)$ , must be also in  $D$ . Thus they are diagonal matrices with 1's as the only permissible nonzero entries.*

*Proof.* This follows from Proposition 2.10 and the fact from the proof of Proposition 2.11 that  $\Delta(N) = D$ .  $\square$

**Remark.** Note that the natural approximate identity for  $\Delta(A) = D_0$  is not an approximate identity for  $A$  (since  $D_0 A \subset A_0 A \subset A_0 \neq A$ ). Thus  $A$  is not  $\Delta$ -dual in the sense of [3]. By [13] we know that  $A$  has an approximate identity which is ‘positive’ in a certain sense.

We recall that an *r-ideal* in  $A$  is a right ideal with a left cai, and an *ℓ-ideal* is a left ideal with a right cai. These objects are in bijective correspondence with the open projections in  $A^{**}$ . Indeed, the limit of such one-sided cai in  $A^{**}$  exists, and is an open projection in  $A^{**}$  called the *support projection* of the one-sided ideal. Conversely, if  $p$  is an open projection in  $A^{**}$  then  $\{a \in A : pa = a\}$  is an r-ideal (and replacing  $pa$  here by  $ap$  gives an ℓ-ideal).

We recall from [3] that  $A$  is *nc-discrete* if all the open projections in  $A^{**}$  are also closed (or equivalently, as we said above, lie in the multiplier algebra  $M(A)$ ). In [2, p. 76] we asked if every approximately unital (semisimple) operator algebra which is a modular annihilator algebra, is weakly compact, or is nc-discrete in the sense of [3]. In [2] we showed that any operator algebra which is weakly compact is nc-discrete. To see that our example  $A$  is not nc-discrete note that  $A_0$  is an r-ideal in  $A$  (and an ℓ-ideal), and its support projection  $p$  in  $A^{**}$ , which is central in  $A^{**}$ , coincides with the support projection of  $D_0$  in  $A^{**}$ , and this is an open projection in  $A^{**}$  which we will show is not closed.

**Corollary 2.13.** *The algebra  $A$  above is not nc-discrete.*

*Proof.* We saw that  $p$  above was open. If  $p$  also was closed in  $A^{**}$ , or equivalently in the multiplier algebra  $M(A)$ , then  $\tilde{\pi}(1-p)$  would be a nonzero central projection



in the copy of  $M(A)$  in  $M$ . Also  $\tilde{\pi}(1-p)e_{i,i}^n$  is nonzero for some  $n$  and  $i$ , because the strong operator topology sum of the  $e_{i,i}^n$  in  $M$  is 1. On the other hand, since  $e_{i,i}^n$  is in the ideal supported by  $p$  we have

$$\tilde{\pi}(p)e_{i,i}^n = \tilde{\pi}(pe_{i,i}^n) = \tilde{\pi}(e_{i,i}^n) = e_{i,i}^n,$$

and so

$$\tilde{\pi}(1-p)e_{i,i}^n = \tilde{\pi}(1-p)\tilde{\pi}(p)e_{i,i}^n = 0.$$

This contradiction shows that  $A$  is not nc-discrete.  $\square$

Indeed  $A_0$  is a nice  $r$ - and  $\ell$ -ideal in  $A$  which is supported by an open projection which is not one of the obvious projections, and is not any projection in  $M(A)$ . Note that  $A$  is not a left or right annihilator algebra in the sense of e.g. [12, Chapter 8], since for example by [12, Chapter 8] this implies that  $A$  is compact, whereas above we showed that  $A$  is not even weakly compact. The spectrum of  $A$  is discrete, and every left ideal of  $A$  contains a minimal left ideal, by [12, Theorem 8.4.5 (h)]. Also every idempotent in  $A$  belongs to the socle by [12, Theorem 8.6.6], hence to  $A_{00}$  by the next result. From this it is clear what all the idempotents in  $A$  are.

**Corollary 2.14.** *The maximal modular right (resp. left) ideals in  $A$  are exactly the ideals of the form  $(1-e)A$  (resp.  $A(1-e)$ ) for a minimal idempotent  $e$  in  $A$  which is the canonical copy in  $A$  of a minimal idempotent in  $M_n$  for some  $n \in \mathbb{N}$ . The socle of  $A$  is  $A_{00}$ , namely the set of  $(a_n) \in A$  with  $a_n = 0$  except for at most finitely many  $n$ .*

*Proof.* Let  $e = (e_n)$  be a (nonzero) minimal idempotent in  $A$ . Then  $e_n$  is an idempotent in  $M_n$  for each  $n$ . If  $e_{i,i}^n$  is as above, then because the strong operator topology sum of the  $e_{i,i}^n$  in  $M$  is 1, we must have  $ee_{i,i}^ne \neq 0$  for some  $n$  and  $i$ . Since  $e$  is minimal, for such  $n$ ,  $e$  is in the copy of  $M_n$  in  $A_0$ . So this  $n$  is unique, and  $e$  is clearly a minimal idempotent in this copy of  $M_n$  in  $A_0$ . Now it is easy to see the assertion about the socle of  $A$ . By [12, Proposition 8.4.3], it follows that the maximal modular left ideals in  $A$  are the ideals  $A(1-e)$  for an  $e$  as above. We have also used the fact here that  $A$  has no right annihilators in  $A$ . Similarly for right ideals.  $\square$

**Corollary 2.15.** *The only compact projections (in the sense of [6]) in  $A^{**}$  for the algebra  $A$  above are the obvious ‘main diagonal’ ones; that is the projections in  $D_0 \cap A_{00}$ .*

*Proof.* Let  $T = (T_n) \in A$ , and  $\epsilon \in (0, \frac{1}{4p(T)})$  be given. As in the proof of Corollary 2.9 there exists  $m_0 \in \mathbb{N}$  such that if  $|\lambda| > 4p(T)\epsilon$  then  $\lambda I - T_n$  is invertible for  $n \geq m_0$ , and the usual matrix norm of its inverse is bounded independently of  $n \geq m_0$ . As in that proof, if  $S_n = T_n$  for  $n \geq m_0$ , and  $S_n = 0$  for  $n < m_0$ , then  $\lambda I - S$  is invertible in  $N$ . Thus the spectral radius  $r(S) \leq 4p(T)\epsilon < 1$ . Hence  $\lim_{k \rightarrow \infty} S^k = 0$  in norm. Let  $q$  be the central projection in  $A$  corresponding to the identity of  $\bigoplus_{n=1}^{m_0-1} M_n$ . If now also  $T \in \frac{1}{2}\mathfrak{F}_A$ , then  $T^k$  converges weak\* to its peak projection  $u(T)$  weak\* by [6, Lemma 3.1, Corollary 3.3], as  $k \rightarrow \infty$ . Thus  $T^k q \rightarrow u(T)q$  and  $T^k(1-q) = S^k \rightarrow u(T)(1-q)$  weak\*. Clearly it follows that  $u(T)q$  is a projection in  $A$ , hence in  $D_0 \cap A_{00}$  as we said above. On the other hand, since  $S^k \rightarrow 0$  we have  $u(T)(1-q) = 0$ . Thus  $u(T)$  is a projection in  $D_0 \cap A_{00}$ .

Finally we recall from [6] that the compact projections in  $A^{**}$  are decreasing limits of such  $u(T)$ . Thus any compact projection is in  $D_0 \cap A_{00}$ .  $\square$

One may ask if there exists a *commutative* semisimple approximately unital operator algebra which is a modular annihilator algebra but is not weakly compact. Later, after this paper was submitted we were able to check that the algebra constructed in [9] was such an algebra. However this example is quite a bit more complicated than the interesting noncommutative example above.

### 3. A COMPLEMENTARY EXAMPLE

In [2, p. 76] we asked if for an approximately unital commutative operator algebra  $A$ , which is an ideal in its bidual (or equivalently that multiplication by any fixed element of  $A$  is weakly compact), is the spectrum of every element at most countable; and is the spectrum of  $A$  scattered? In particular, is it a modular annihilator algebra (we recall that compact semisimple algebras are modular annihilator algebras [12, Chapter 8]). There is in fact an easy semisimple counterexample to these questions, which is quite well known in other contexts. The algebra  $A$  will in fact be unital and isomorphic to a Hilbert space, so is Banach space reflexive, hence is obviously an ideal in its bidual. It is also singly generated by an operator  $T$ , and the identity  $I$ , so that by basic Banach algebra theory the spectrum of  $A$  is homeomorphic to  $\text{Sp}_A(T)$ . The example may be described either in the operator theory language of weighted unilateral shifts, and the  $H^p(\beta)$  spaces that occur there, or in the Banach algebra language of weighted convolution algebras  $l^p(\mathbb{N}_0, \beta)$ . These are equivalent (in particular,  $H^2(\beta) = l^2(\mathbb{N}_0, \beta)$ ). We begin with the Banach algebraic angle: The weighted convolution algebras  $l^1(\mathbb{N}_0, \beta)$  are much studied (see e.g. [10]), and they are Banach algebras whenever the weight  $\beta$  is an “algebra weight”, i.e.  $\beta_{i+j} \leq \beta_i \beta_j$  for all  $i, j$ . Sometimes, moreover, the weighted  $l^2$  space  $l^2(\mathbb{N}_0, \beta)$  is a Banach algebra under the convolution product, and in such cases it is an operator algebra that is isomorphic (as Banach space) to a Hilbert space. One such case is the weight  $\beta_n = C(1+n)$  for suitable  $C > 1$ . In any such case the generator acts on  $l^2(\mathbb{N}_0, \beta)$  as a weighted shift operator which is unitarily equivalent to a weighted shift on  $l^2(\mathbb{N}_0)$  with weights  $w_i = \beta_{i+1}/\beta_i$ .

From the operator theory angle, in the 1960’s and 70’s, operator theorists exhaustively studied weighted shifts and the algebras they generate. See e.g. Shields’ 1974 survey [14] for this and the details below. Let  $T$  be a weighted unilateral shift which is one-to-one (that is, none of the weights  $w_n$  are zero), and let  $A$  be the algebra generated by  $T$ . Then  $A$  is isomorphic to a Hilbert space if  $T$  is *strictly cyclic* in Lambert’s sense [14], that is there is a vector  $\xi \in H$  such that  $\{a\xi : a \in A\} = H$ . Central to the theory of weighted shifts is the convolution algebra  $H^2(\beta) = l^2(\mathbb{N}_0, \beta)$ , and its space of ‘multipliers’  $H^\infty(\beta)$ . These spaces can canonically be viewed as spaces of converging (hence analytic) power series on a disk, via the map  $(\alpha_n) \mapsto \sum_{n=0}^\infty \alpha_n z^n$ . Here  $\beta$  is a sequence related to the weights  $w_n$  above by the formula  $\beta_n = w_0 w_1 \dots w_{n-1}$ . For example, one such sequence is given by  $\beta_n = n+1$ , an example mentioned in the last paragraph, and the spectral radius of the weighted shift here is 1. By the well known theory in [14],  $T$  is unitarily equivalent to multiplication by  $z$  on  $H^2(\beta)$ , the latter viewed as a space of power series on the disk of radius  $r(T)$ . In our strictly cyclic case,  $A$ , which equals its weak closure, is unitarily equivalent via the same unitary to  $H^\infty(\beta)$ . Since  $H^\infty(\beta)H^2(\beta) \subset H^2(\beta)$  and the constant polynomial is in  $H^2(\beta)$ , it is clear that  $H^\infty(\beta) \subset H^2(\beta)$ . However, since the constant polynomial 1 is a strictly cyclic vector, we in fact have  $H^\infty(\beta) = H^\infty(\beta)1 = H^2(\beta)$  (see p. 94 in [14]). On the same

page of that reference we see that the closed disk  $D$  of radius  $r(T)$  is the maximal ideal space of  $H^\infty(\beta)$ , and the spectrum of any  $f \in H^\infty(\beta)$  is  $f(D)$ . In particular,  $\text{Sp}_A(T) = D$ , and  $A$  is semisimple. We remark in connection with a discussion with Dales, that this implies that  $A$  is a natural Banach function algebra on the disk, but it is not a Banach sequence space in the sense of Section 4.1 in [10], since  $A$  contains no nontrivial idempotents (since  $A$  may be viewed as analytic functions on a disk).

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